Methodological Note
Producing the Irish Life Table, No. 16
2010-12

Kevin McCormack: Central Statistics Office Ireland

Abstract

In this paper the most recent statistical methodology used to produce the Irish Life Table, No. 16, (ILF16), covering the period 2010-12 is described. Crude mortality rates are smoothed, or graduated, using a modern and more statistically accurate self-adaptive free-knot cubic-spline graduation method based on the B-Spline methodology. Life tables for males and female are constructed.

Keywords: Mortality: B-Spline; Knots; Graduation

1. Introduction

A life table is a convenient way of summarising various aspects of the variation of mortality with age and gender in the three-year period around a census year. The graduation or smoothing of crude population mortality rates is essential in the construction of life tables as the recording of the underlying deaths are subject to errors.

Period Life Tables have been produced by the Irish Central Statistics Office (CSO) on fifteen occasions, from 1926 to 2005-07, and on each occasion the King's 1911 formula for Osculatory Interpolation was used to graduate the crude mortality rates.

In this paper, the statistical methodology underlying the production of ILT16, the 16th version of the Irish Life Tables is described.

2. The Measurement of Morality

2.1 The crude death rate – central mortality rate

The simplest measure of mortality is the number of deaths. However, this measure is heavily influenced by the number of people who are at risk-of-dying.

Because of this, mortality is measured using *rates*. A death rate is defined as

$$Death\ Rate\ =\ \frac{\text{number of deaths in a specified time period}}{\text{number of people exposed to the risk of dying during that time period}}$$

Thus, in order to measure mortality, data are required about the number of people exposed to the *risk-of-dying*. Data on the number of people exposed to the *risk of dying* are usually obtained from a census of population.

The most straight forward death rate is the total number of deaths in a given time period divided by the total population. This measure is called the *crude death rate*. The time period used is normally one calendar year. Thus,

$$Crude\ Death\ Rate\ =\ \frac{\text{total number of deaths in a given year}}{\text{total population}}$$

An immediate issue arises with the measurement of the total population. During any year, the population will usually change. At what point in the year, therefore, should it be measured? Conventionally, the point chosen is half-way through the year (30 June). The population on 30 June is called the *mid-year population*. Using this definition of the population exposed to the risk of dying, therefore,

$$Crude\ Death\ Rate\ =\ \frac{\text{total number of deaths in a given year}}{\text{total mid} - \text{year population}}$$

Denoting the crude death rate in year t by the symbol d_t , the total number of deaths in year by θ_t, and the total population on 30 June in year t by the symbol P_t, we can write

$$d_t = \frac{\theta_t}{P_t} \qquad (2.1)$$

For ease of presentation, the subscripts t are usually omitted because, unless otherwise stated, the period of time over which the crude death rate is measured may be assumed to be a single calendar year. Thus

$$d = \frac{\theta}{P} \qquad (2.2)$$

2.2 Age-specific death rates

The crude death rate does not provide a great deal of information about mortality. In particular, the risk of dying varies greatly with age, and the crude death rate indicates nothing about this variation. Because of this *age-specific death rates* are used.

The age-specific death rate at age x is defined as

$$\text{Age} - \text{specific death rate at age } x = \frac{\text{number of deaths of people aged } x}{\text{population aged } x \text{ years}}$$

in a given calendar year. When we refer to 'age x years', we mean 'aged x last birthday. The denominator, as before, is the mid-year population.

Denoting,

m_x the age-specific death rate at 'age x years', also known as the central mortality rate,

θ_x number of deaths of people age x years last birthday, and

P_x the population aged x years last birthday,

we can write

$$m_x = \frac{\theta_x}{P_x} \qquad (2.3)$$

Note that x denote years of age, not calendar years.

3 Graduating (Smoothing) – removing errors in population and mortality data

It is accepted that both the population and mortality data used in the calculation of m–type mortality rates are subject to error. These errors take two well-defined forms:

i. errors in the age that members of a population record on their census returns
ii. errors in the age reported at the death of a member of a population.

Therefore, one can view the underlying data that are used in the calculation of m–type mortality rates as being made up of two component parts:

$$\text{Data} = \text{Smooth} + \text{Rough} \qquad (3.1)$$

The 'rough' part - or error – is due to the measurement and sampling errors outlined above.

In addition to mortality estimation, it is necessary to correct for these errors in order to provide a more accurate estimate of the age distribution of the population.

Graduating, or smoothing as it also known, is the application of specific methods to remove errors from data.

These is a saying "natura on agit per saltum" expressing the fundamental fact that natural forces operate gradually and that their effects become apparent continuously and not in sudden jerks. In its application to mortality data it implies that any rates which may reflect the operation of purely natural causes should not exhibit any discontinuities, breaks, or sudden and unexpected changes. In other words, one expects that any set of true values to follow a smooth curve, or that the graduated series must possess a high degree of smoothness.

For practical purposes the table of m–type mortality rates which are to be extensively used should have a very high degree of smoothness: otherwise the more complicated function based on it, such as insurance policy values, will show alarming and even embarrassing irregularities.

3.1 Change of Methodology

Period Life Tables have been produced by the Irish Central Statistics Office (CSO) on fifteen occasions, from 1926 to 2005-07, and on each occasion the King's 1911 formula for Osculatory Interpolation was used to graduate the crude mortality rates. King's method of osculatory interpolation is an example in which cubic curves are fitted together to produce a function which is everywhere differentiable.

It is important, however, to note that in general King's method determines a function which has only *one* derivative at the points where the adjacent cubics join. This method has the least general application internationally, falling out of favour in the 1930s. The method does not graduate crude death rates since ungraduated rates are not calculated. Since population numbers and deaths are graduated separately, the method is also susceptible to features that affect population numbers but not the mortality rates, such as fluctuations in numbers of births.

Smoothing has become part of the standard statistical toolbox, and the facility to include smooth functions of covariates into a regression model is included in standard software. A smooth term in a regression model is typically included as a cubic spline, which is a piecewise polynomial, with a number of knots (i.e. points where they join). Splines with no knots are generally smoother than splines with knots, which are generally smoother than splines with multiple discontinuous derivatives. Splines with few knots are generally smoother than splines with many knots; however, increasing the number of knots usually increases the fit of the spline function to the data. One must note, however, that the regression curve must not be overly parametrised, (i.e., it does not include too many knots and regression coefficients).

The use of cubic splines provides greater smoothness than that provided by King's method, since a second derivative exists everywhere. In addition, given the widespread adoption of the cubic spline approach for smooth function estimation, the use of a cubic spline is viewed as a natural approach for the graduation of the Irish Life Table No. 16. In particular, a B-Spline model is applied.

In Section 4, the ungraduated mortality data is for Ireland 2010-2012 are presented, and the smoothing B-spline graduation is illustrated. In Section 5, the data analysis and the final life tables are presented, together with conclusions, in Section 6.

4. General Data Analysis

The crude mortality rates (on a logarithmic scale) are presented in *Figure 1*. Several features are immediately apparent. As would be expected, male mortality rates are higher than female mortality rates throughout the age range, but with some convergence at older ages. Mortality decreases throughout the first few years of life, with a particularly steep drop between ages 0 and 1. From about age 10 onwards mortality increases, with a particularly steep increase in late teenage years, particularly pronounced for males, and attributable to a higher rate of death from accidents (sometimes referred to as the 'accident hump').

The final feature of note, of particular interest here, is the fact that crude mortality rates exhibit much greater variability in the highest (over 100) age groups.

The life table has been constructed using the graduated mortality rates m_x, $x = 0, \ldots, 105$, in order to calculate the values of q_x (probability of death at age x). For this purpose, the following methodology, proposed by McCutcheon (1975-77), has been applied.

4.1 Mortality at Young Ages (0-2 Years)

Special techniques were used for the measurement of mortality at ages 0 and 1. It is important to note that over the first two years of life the mortality function, l_x (the number of a cohort who live to experience their x^{th} birthday), is unlikely to resemble a quadratic curve. For this reason the values of q_0 and q_1 have been derived directly from the data on births and infant deaths.

To obtain m_0 the following formula was applied:

$$m_0 \approx \frac{q_0}{1 + (1 - \phi_0) q_0} \qquad (4.1)$$

where ϕ_0 = the average age at death of those who die in the first year of life,

m_0 = the crude mortality rate at age 0 and

$$q_0 = \frac{\theta_0}{E_0}$$

To obtain m_1 the following formula was applied:

$$m_1 = q_1 \left[\frac{1 + \frac{5}{12} m_2}{1 + \left(\frac{1}{2} - \frac{1}{3} q_1 \right) m_2 - \left(\frac{7}{12} \right) q_1} \right] \qquad (4.2)$$

where m_1 = the graduated death rate at age 1 year, and

m_2 = the graduated death rate at age 2 year.

$$q_2 = m_2 \left[\frac{1 + \frac{13}{12} q_1}{(1 - q_1)\left(1 + \frac{5}{12} m_2 \right)} \right] \qquad (4.3)$$

Equations 4.1, 4.2 and *4.3* valid under the assumption that l_x is quadratic over the age range $[1, 3]$.

4.2 Mortality at Ages 3 and above

In order to convert the graduated central mortality rates m_x into q_x values, for ages $x = 3, \ldots, 100$, the following approximation was applied:

$$q_x \approx m_x \left[\frac{1 + \frac{1}{2} m_{x-1}}{1 + \left(\frac{5}{12} \right)(m_x - m_{x-1}) - \left(\frac{1}{6} \right) m_x m_{x-1}} \right]$$

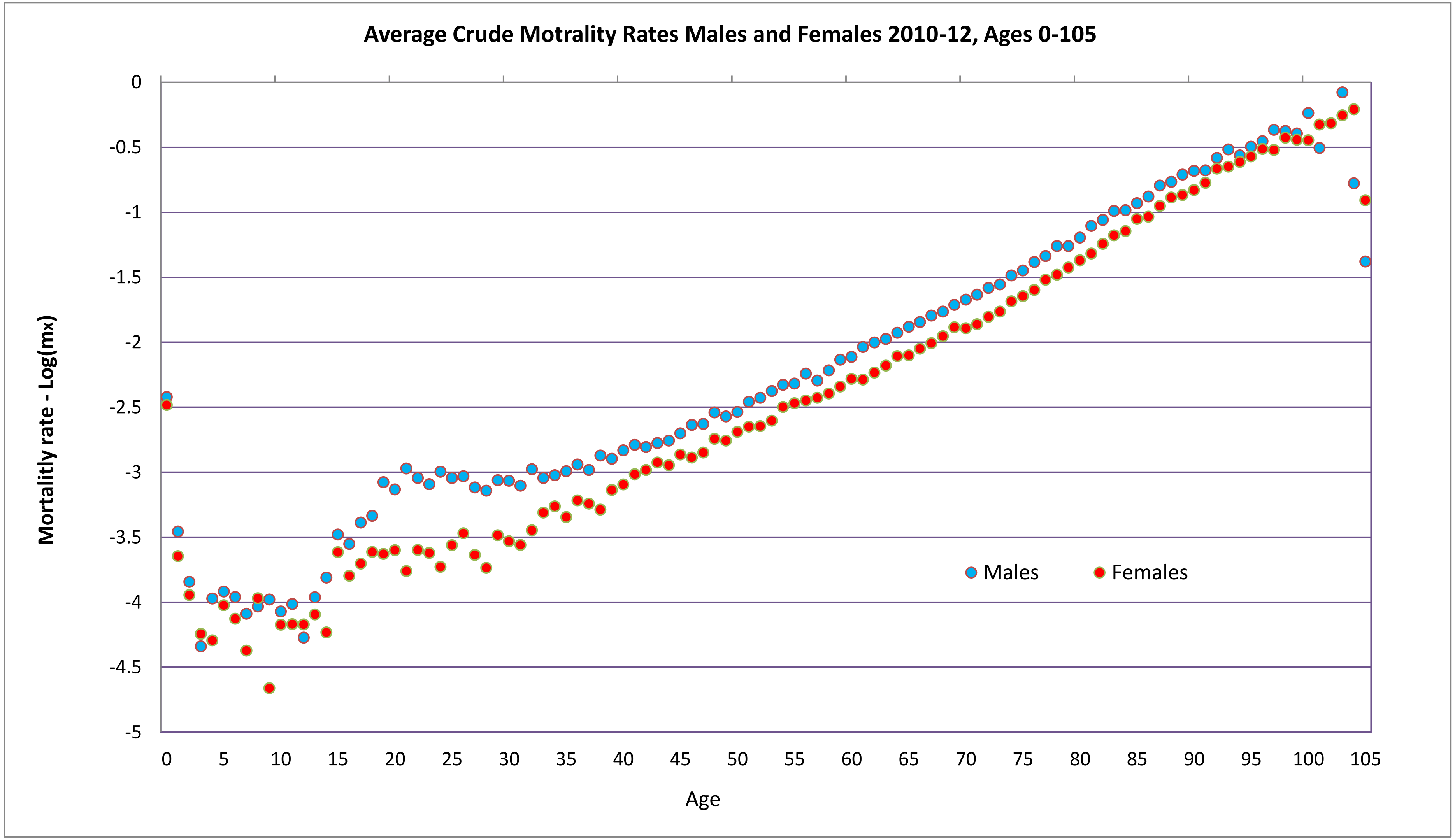

Figure 1: Average crude mortality rates for Males and Females, 2010-12.

4.3 Performance of self-adaptive free-knot cubic B-Spline regression model

The performance of least-squares splines are dependent upon the number and location of the knots for the polynomial segments. In particular, when the number of knots is small, proper knot location becomes paramount for obtaining acceptable results. Optimal knot placement is a nonlinear problem with a known lethargy property, as described by Jupp (1975), that does not readily lend itself to derivative-based optimization methodology.

The crude mortality rates where graduated, using a self-adaptive free-knot cubic-spline method based on the B-Spline methodology which addresses the lethargy property. There are two parts to the graduation process: firstly the knot locations are identified, and subsequently included in a cubic B-Spline regression model. The method is discussed in more detail in *Section 6*.

First observations on both the male and female models are that over the majority of ages the graduated curves is reasonably smooth and at the same time follows acceptably well to the crude mortality rates – it can be said to "*pass the eye-ball test*". At all ages the graduated mortality rates males are higher than those for females. This indicates that the model is correctly reflecting the diverse demographic features in the underlying data.

The curve is less smooth at the early ages (i.e. 0-3 years) for both males and females and at the older ages: 100-105 years for males and 104-105 years for females. However, as previously observed, the graduated curves follow acceptably well the crude mortality rates in these areas of high variability.

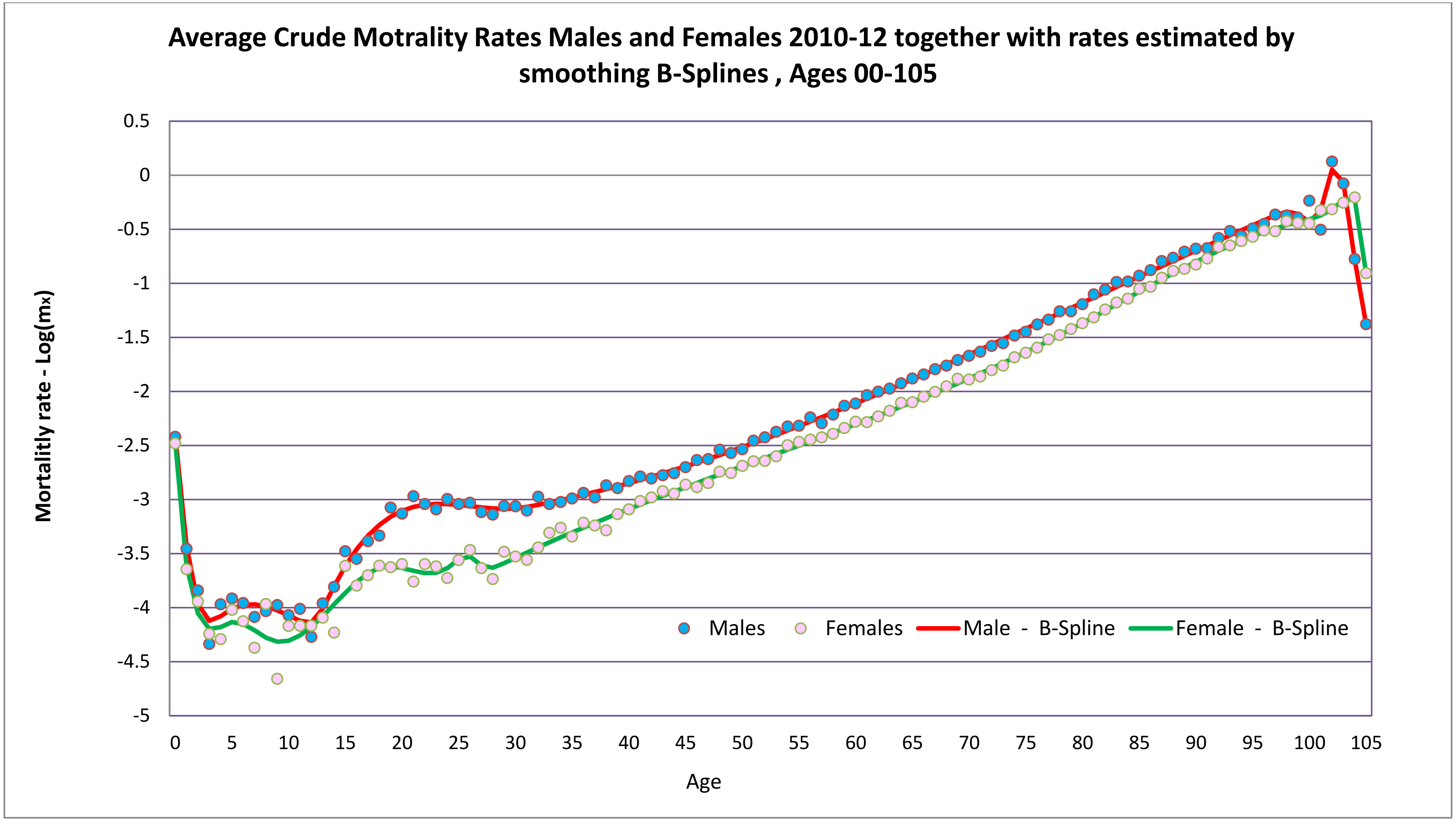

Figure 2: Average crude mortality rates for Males and Females, 2010-12 together rates estimated by smoothing B-Splines

5. Smoothing Irish Crude Morality Rates using cubic- B-SPLINE methodology

5.1 Introduction

Splines are curves, which are required to be continuous and smooth. Splines are usually defined as piecewise polynomials of degree *n* with function values and first *n-1* derivatives that agree at the points where they join. The abscissa values of the join points are called ***knots***. The term "spline" is also used for polynomials (splines with no knots) and piecewise polynomials with more than one discontinuous derivative. Splines with no knots are generally smoother than splines with knots, which are generally smoother than splines with multiple discontinuous derivatives. Splines with few knots are generally smoother than splines with many knots; however, increasing the number of knots usually increases the fit of the spline function to the data. Knots give the curve freedom to bend to more closely follow the data. A detailed description of splines is given by De Boor (1978).

5.2 Description of the cubic-spline method – *B*-spline

In the specific case of the construction of life tables, the observations are age points $\{x_1, \dots, x_N\}$ in the range $[a, b]$, satisfying $[a < x_1 < \dots < x_N < b]$ and the crude rates of mortality (denoted by m_x), the y-values, at these age points. In the case of the CSO's Irish Life Table No. 16, the m_xs were constructed by dividing the average number of deaths at each age in each of the three calendar years 2010, 2011 and 2012, (A_x), by an exposed-to-risk from the 2006 Census of Population, denoted by (P_x), or (E_x^c), $\left[i.e., m_x = \frac{A_x}{E_x^c}\right]$.

Crude death rates are not normally used when constructing life tables, because they tend to fluctuate unpredictably for one age to another when small numbers of deaths are recorded, most notably at very young and advanced ages. The errors arising due to the small number of deaths can be reduced by the process of smoothing the crude death rates.

The first step in the smoothing procedure is to apply an logarithmic transformation of the crude mortality rates

$$y_i = \log(m_x), \quad \text{i} = 1, 2, \dots, \text{N} \qquad (5.1)$$

It is then assumed that there is an (unknown) functional relationship between the (response) variable y and the age variable x of the form

$$y = f(x) + \varepsilon, \qquad (5.2)$$

where

- ε is a random (observation) error variable with zero mean and some variance σ^2, and
- (f) is an unknown function, approximated with a n^{th} order (degree $n - 1$) polynomial spline.

A spline function $f(\mathbf{t}_{k,n}; x)$ on an interval $[a, b]$, consists of pieces of polynomials of a certain degree, $n - 1$, and these pieces are smoothly jointed together at some points

$$\mathbf{t}_{k,n} = \{t_1 = \cdots = t_n = a < t_{n+1} < \cdots < t_{n+k} < t_{n+k+1} = b = \cdots = t_{2n+k}\}$$

called the **knots** of the spline.

The spline function can be represented by

$$f(\mathbf{t}_{k,n}; x) = \boldsymbol{\theta}'\mathbf{N}_n(x) = \sum_{i=1}^{p} \theta_i N_{i,n}(x), \qquad (5.3)$$

where

$\boldsymbol{\theta}' = (\theta_1, \ldots, \theta_p)$ is a vector of (unknown) regression coefficients, and

$\mathbf{N}_n(x) = (N_{1,n}(x), \ldots, N_{p,n}(x))$ and $p=n+k$ are certain basis (spline) functions, known as $\boldsymbol{B}$-splines of order n.

B-splines are splines defined on $t_{k,n}$ through the Mansfield-De Boor-Cox recurrence relation

$$N_{i,1}(t) \quad = \quad \begin{cases} 1 & if \quad t_i \leq t \leq t_{i+1} \\ 0 & \text{otherwise} \end{cases},$$

$$N_{i,n}(t) \quad = \quad \frac{t - t_i}{t_{i+n-1} - t} N_{i,n-1}(t) + \frac{t_{i+n} - t}{t_{i+n} - t_{i+1}} N_{i+1,n-1}(t)$$

from which it can be seen that $N_{i,n}(t) = 0$ for $t \notin [t_i, t_{i+n}]$. In order to emphasize the dependence of the spline regression $f\left(\mathbf{t}_{k,n}; x\right)$ on $\boldsymbol{\theta}$, the notation $f\left(\mathbf{t}_{k,n}, \boldsymbol{\theta}; x\right)$ is used.

In the current context, the spline regression estimation problem is formulated as follows. Based on:

- a sample of observations of the crude mortality rates, $\{y_i\}_{i=1}^{N}$, at the age points $\{x_i\}_{i=1}^{N}$,
- estimate the degree n - 1 of the spline,
- the number of knots, k,
- the set of knots, $t_{k,n}$, and
- the regression coefficients $\boldsymbol{\theta}$,
- so that the estimated spline curve of the crude mortality rates is sufficiently smooth but at the same time captures all the peculiarities of the shape of the functional relationship in Eq. (5.1).

In addition, it is required that the curve is not overly parametrised, (i.e., it does not include too many knots and regression coefficients $\boldsymbol{\theta}$).

One can then define the *error sum of squares* for the nonlinear spline model and the associated data as

$$S(\boldsymbol{\theta}) \; = \; \sum_{i=1}^{n} \left\{ y_i - f\left(\boldsymbol{t}_{k,n},; \boldsymbol{\theta}; x\right) \right\}^2 \qquad (5.4)$$

which is minimised with respect to the parameters, $\boldsymbol{\theta}$**.** The asymptotic properties of least squares spline regression have been considered by Afarwal and Studden (1980), Huang (2003) and Shang and Cheng (2013).

6 Cubic Spline Model

6.1 Recovering the unknown function

An advanced Cubic Spline model is deployed to efficiently recover the unknown function from a set of set of observations, $\{y_i, x_i\}_{i=1}^{N}$.

6.2 Self-Adaptive Free-Knot Cubic B-Spline Model

A self-adaptive cubic B-Spline model is fitted to the data described in Section 4. This model consists of two stages. In the first stage, a continuous genetic algorithm is employed to locate the optimum positions of the knots from all possible sets of knot positions using an Akaike Information Criterion (AIC) as described by Spiriti (2013). This approach address directly the known lethargy property as previously referenced.

In the second stage, the knot locations from the first stage are used to fit a B-spline regression model to the Male and Female data. The knots are free and readily able to cope with rapid change in the underlying model.

The properties of the smoothing B-Spline was then used to derive 95% and 99% confidence intervals (CI) (i.e. upper and lower lines), using Jacknife residuals approach, consistent with Wahba (1983), and one can expect to cover between 95% and 99% of the true (but in practice unknown) values of $f(x)$.

The gender specific (i.e. Male and Female) $\boldsymbol{log(m_x)}$'s and their fitted values for the B-splines and their 95% & 99% CI lines are presented in *Figures 3* & 4. One can note that, for both genders, the majority of data points fit within 95% CIs.

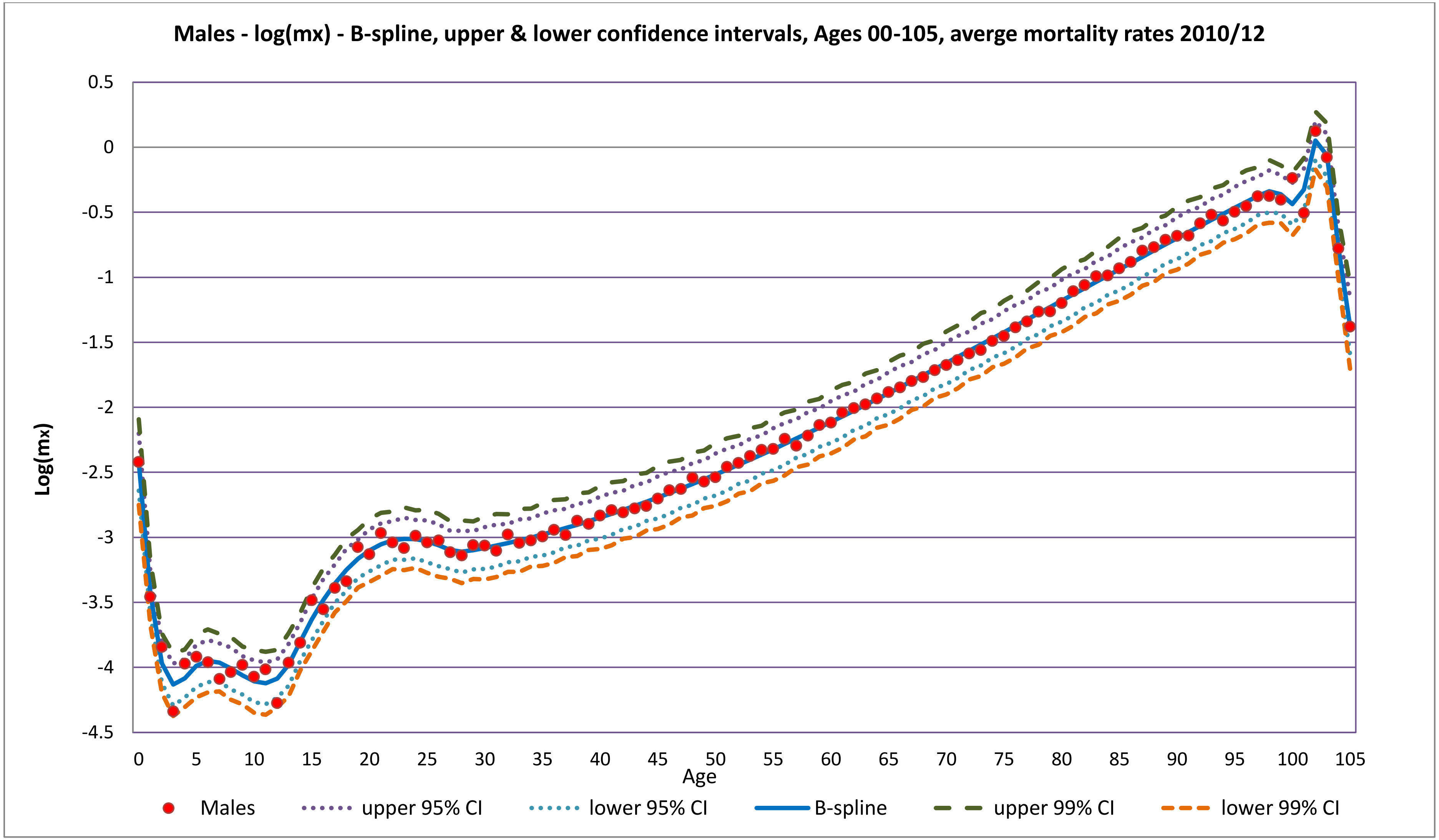

Figure 3: Males - log(mx) - B-spline, upper & lower confidence intervals, Ages 00-105, average mortality rates 2010-2012

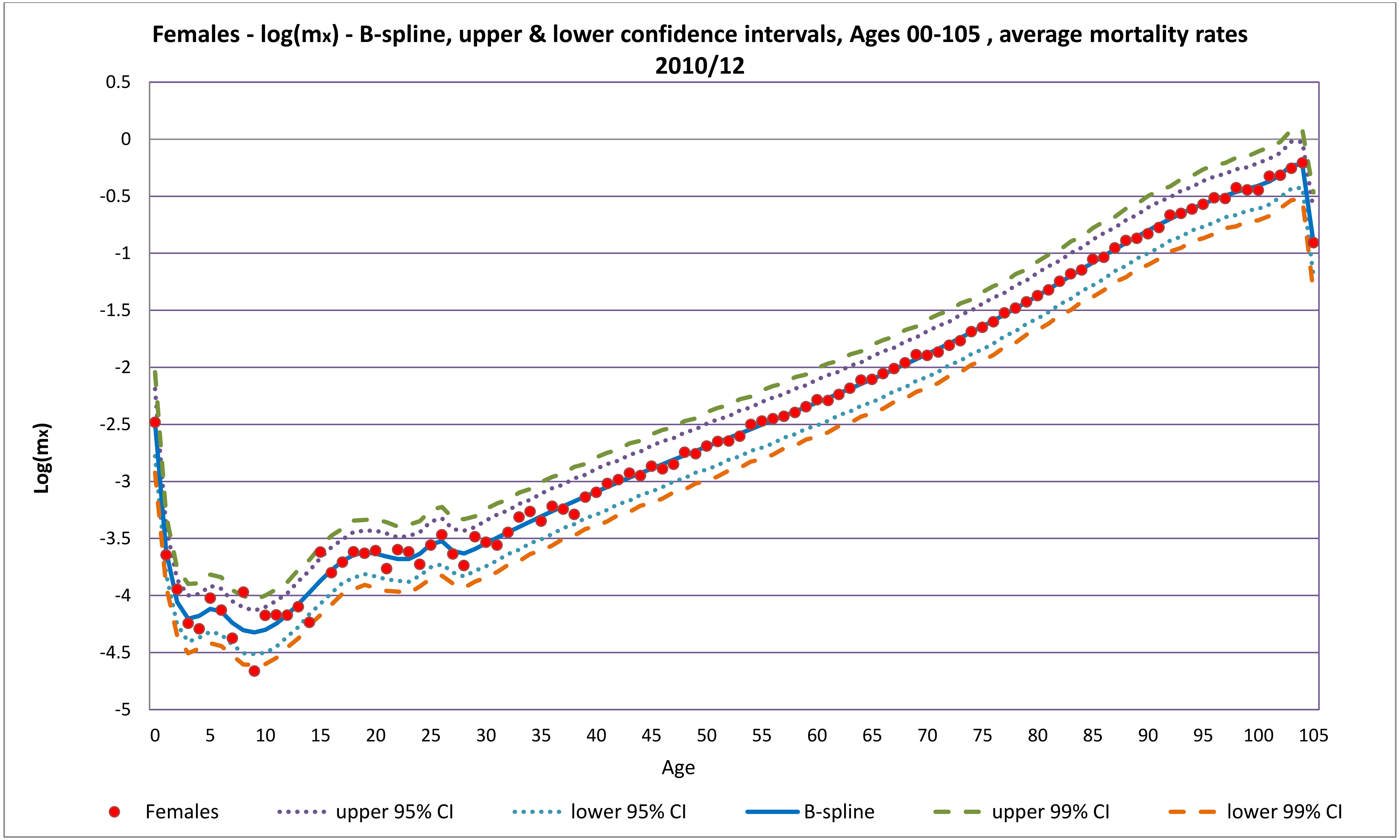

Figure 4: Females - log(mx) - B-spline, upper & lower confidence intervals, Ages 00-105, average mortality rates 2010-2012

6.3 Males - Observations on the fit of the B-Spline model

The transformed age-specific crude death rate at ‘age x years’ (i.e. $\boldsymbol{log(m_x)}$) for males fluctuate considerably under 30 years of age, are linearly increasing between ages 31 and 99 years and then fluctuate again from ages 100 to 105 years. *(See Figure 3).*

6.3.1 Males - Early ages - under 30 years of age

The male the data is quite noisy, with the B-Spline curve closely following the fluctuation for the crude death rates. The curve is generally within the 95% CI except for ages 3 and 12 years, which are within the 99% CI. *(See Figure 5).*

6.3.2 Males - Ages 31 to 69 years

The curve closely follows the linear trend of the crude death rates for males over this age range. The curve is at all times within the 95% CI. *(See Figure 6).*

6.4.3 Males - Ages 70 -105 years

For ages up to 99 years the B-Spline curve closely follows the crude death rates. From age 100 years onwards, the B-Spline curve is less smooth reflecting the variability of the crude death rates at those ages. However, the B-Spline curve remains at all times within the within the 95% CI. *(See Figure 7).*

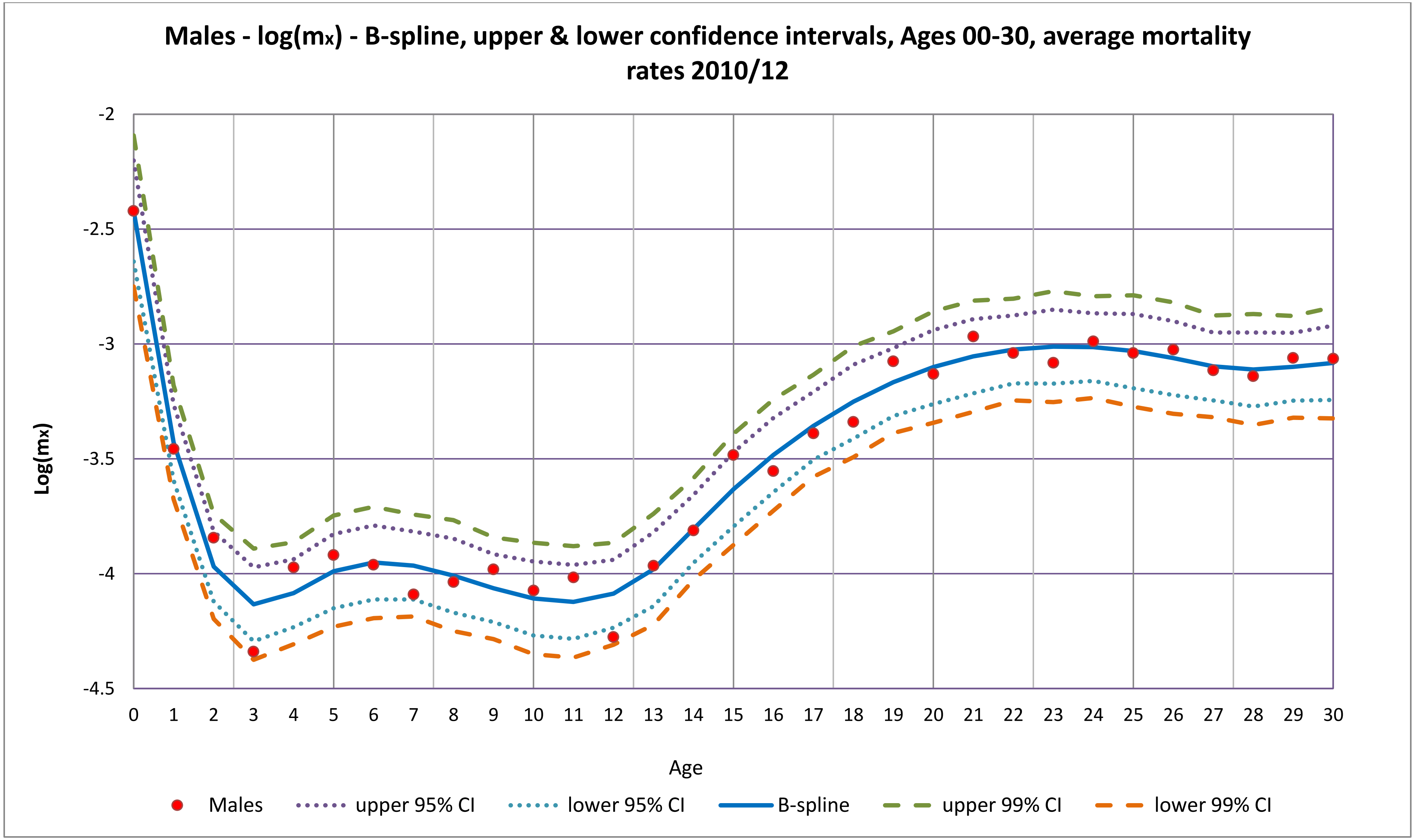

Figure 5: Males - log(mx) - B-spline, upper & lower confidence intervals, Ages 00-30, average mortality rates 2010-2012

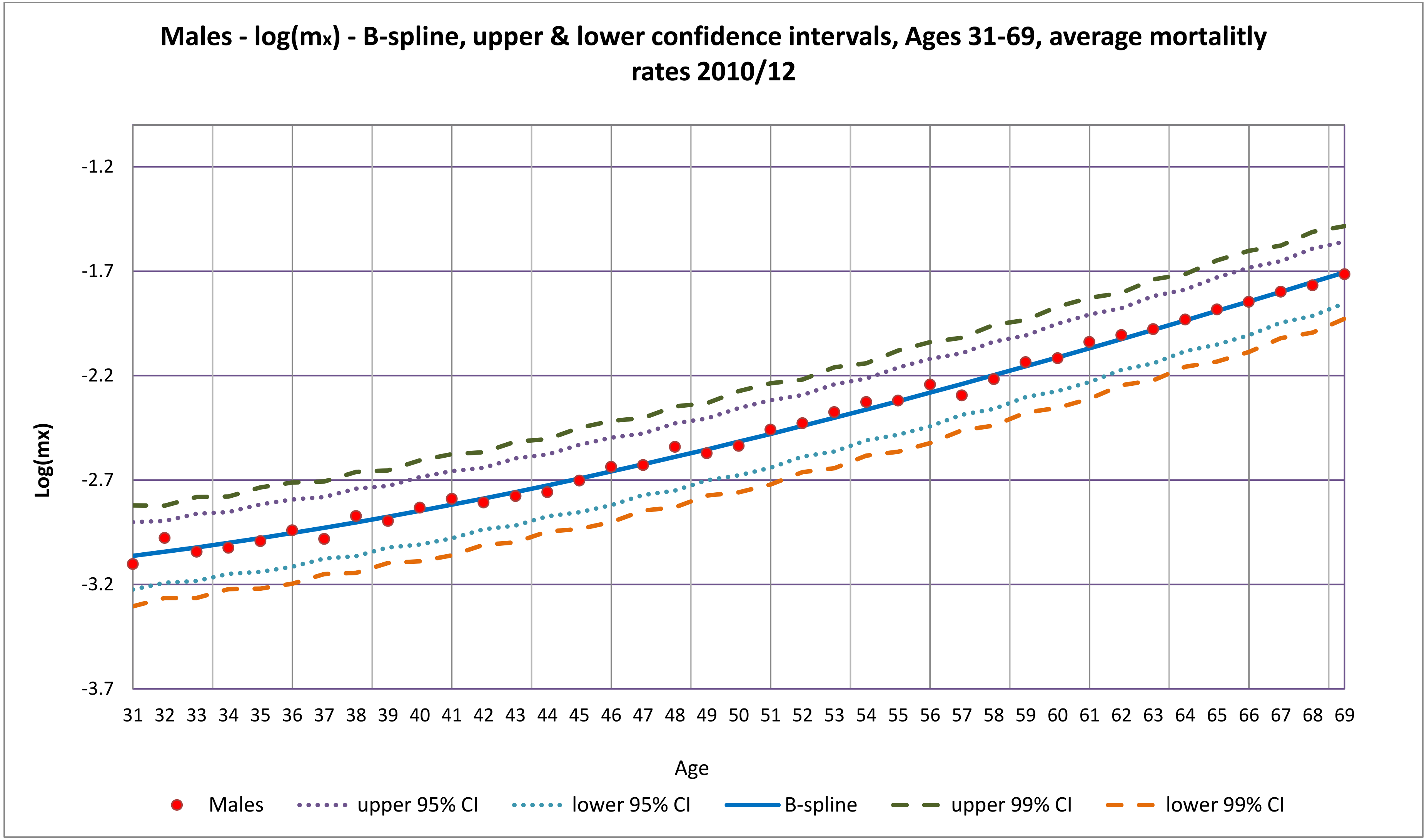

Figure 6: Males - log(mx) - B-spline, upper & lower confidence intervals, Ages 31-69, average mortality rates 2010-2012

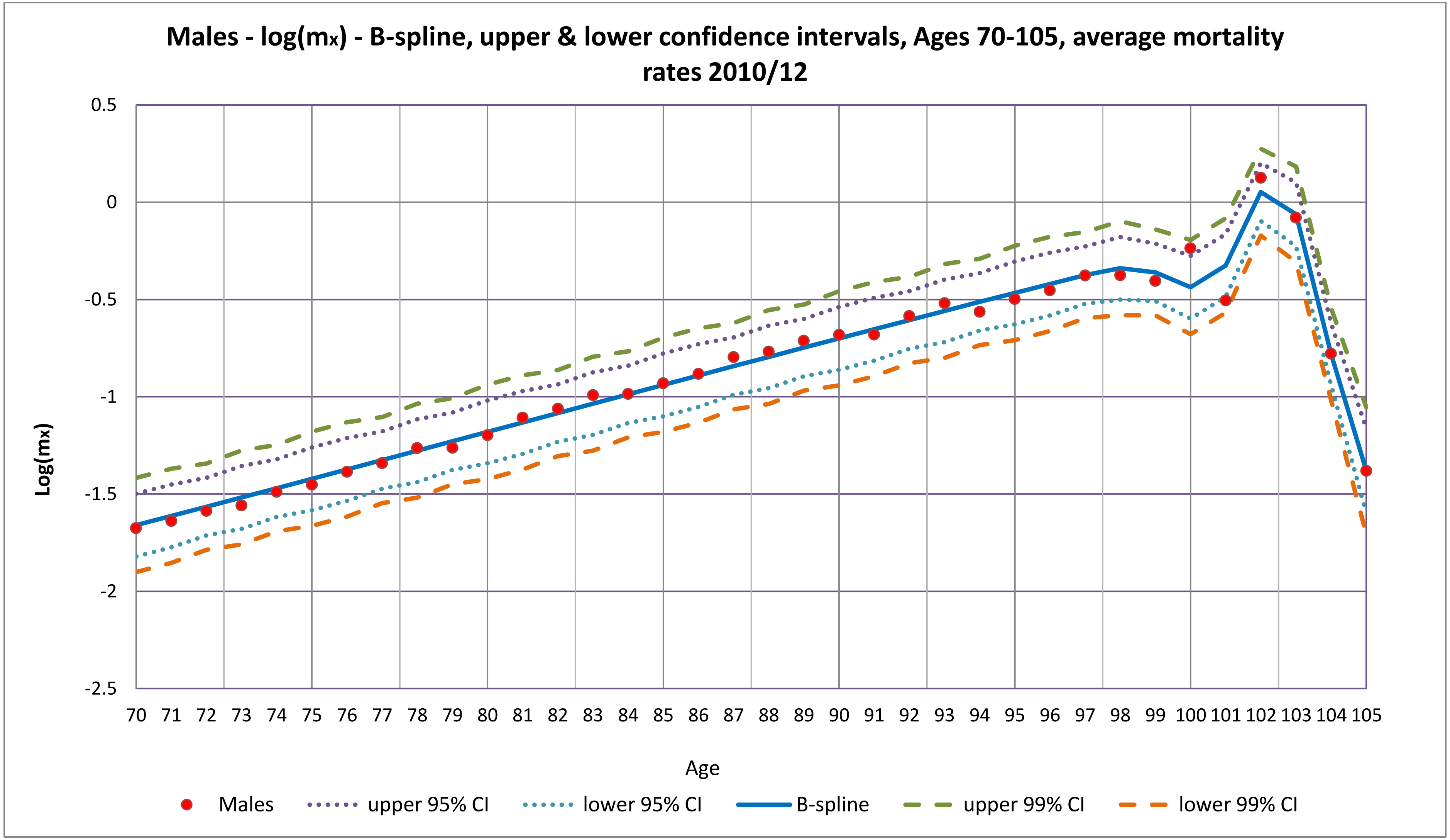

Figure 7: Males - log(mx) - B-spline, upper & lower confidence intervals, Ages 70-105, average mortality rates 2010-2012

6.4 Females - Observations on the fit of the B-Spline model

The transformed age-specific crude death rate at ‘age x years’ (i.e. . $\boldsymbol{log(m_x)}$ for Females fluctuate considerably under 30 years of age, are linearly increasing between ages 31 and 104 and then fluctuate again at age 105. *(See Figure 4).*

6.4.1 Females - Early ages - under 30 years of age

The female data is quite noisy, with the B-Spline curve closely following the fluctuation for the crude death rates. The curve is generally within the 95% CI. However, ages 8 and 9 are outside the 99% CI. There are dramatic movements of the crude mortality rate at these two consecutive ages and B-Spline model provides an appropriate smooth curve between these two age points. *(See Figure 8).*

6.4.2 Females - Ages 31 to 69 years

The curve follows the linear trend of the crude death rates for females over this age range. The curve is at all times within the 95% CI. *(See Figure 9).*

6.4.3 Females - Ages 70 – 99 years

For ages up to 103 years the B-Spline curve closely follows the crude death rates. From age 104 years onwards, the B-Spline curve is less smooth reflecting the variability of the crude death rates at those ages. However, the B-Spline curve remains at all times within the within the 95% CI. *(See Figure 10).*

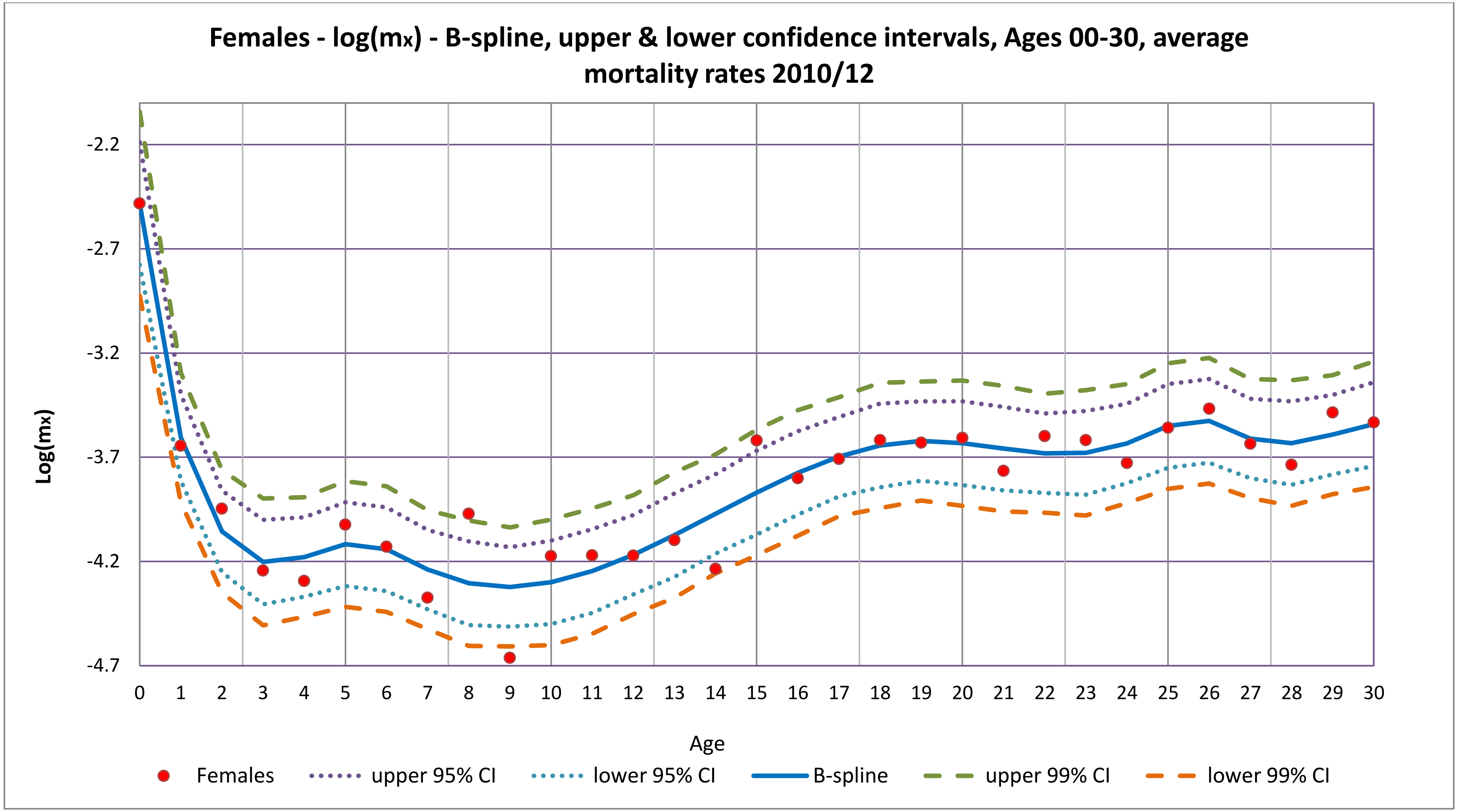

Figure 8: Females - log(mx) - B-spline, upper & lower confidence intervals, Ages 00-30, average mortality rates 2010-2012

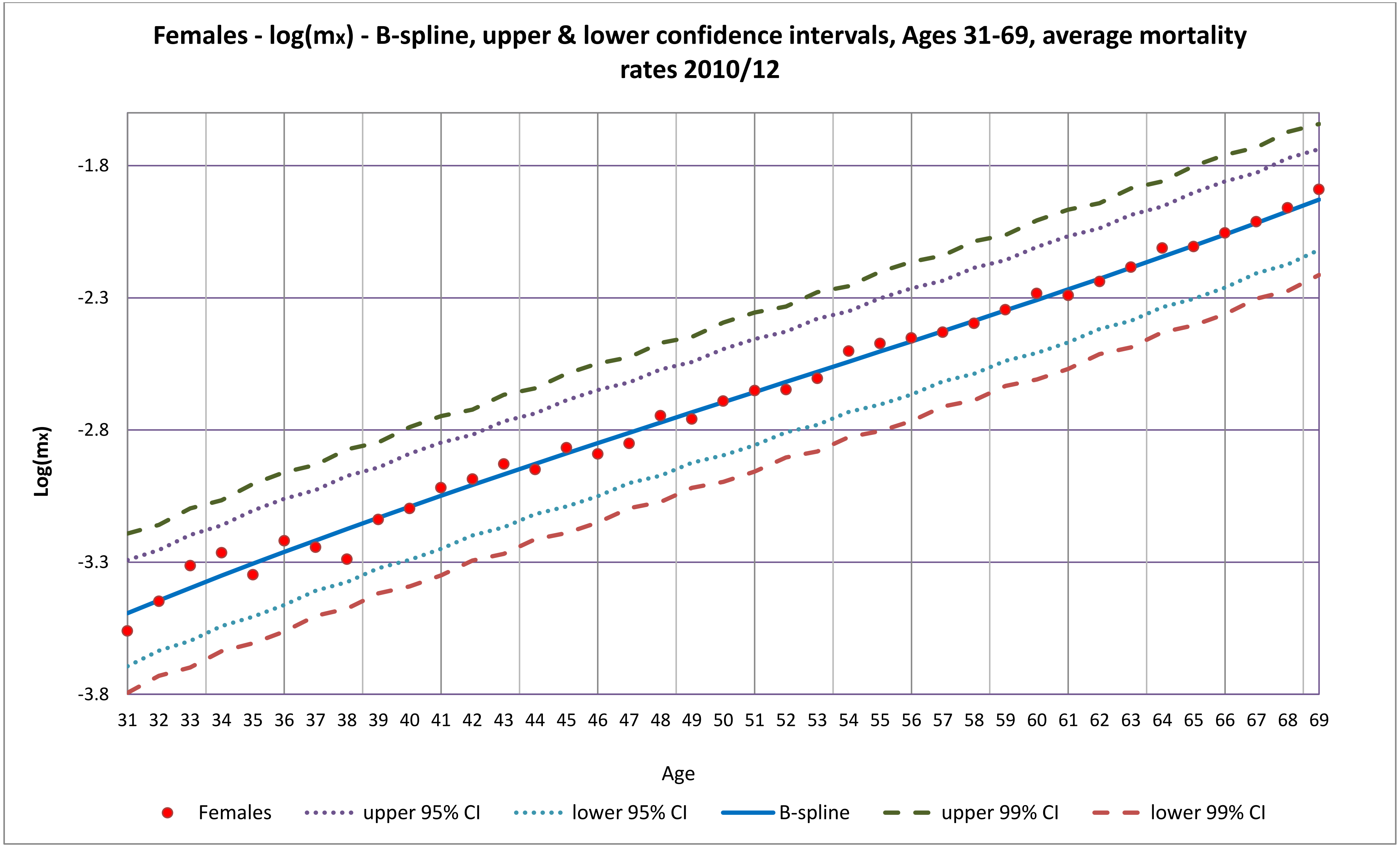

Figure 9: Females - log(mx) - B-spline, upper & lower confidence intervals, Ages 31-69, average mortality rates 2010-2012

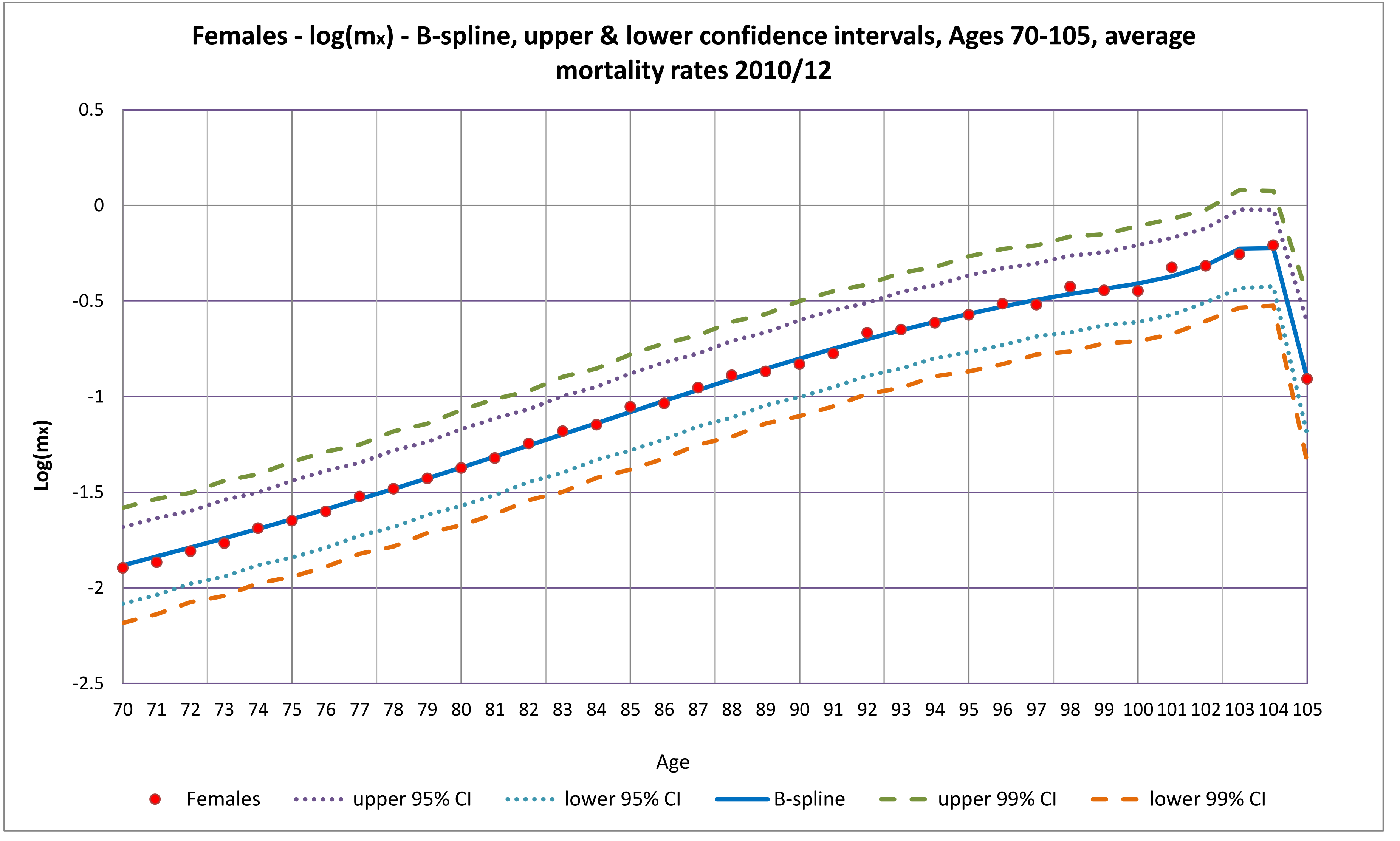

Figure 10: Females - log(m_x) - B-spline, upper & lower confidence intervals, Ages 70-105, average mortality rates 2010-2012

Bibliography

Agarwal, G. G. and Studden, W.J. (1980). *Asymptotic integrated mean square error using least squares and bias minimising splines*. The Annals of Statistics, **8**, 6, 1307-1325.

Department of Health (2016). ***Health in Ireland, Key Trends 2016*****.**

Central Statistics Office, (2009). *The Irish Life Table No 15*.

Central Statistics Office (2011). *Vital Statistics, Fourth Quarter and Yearly Summary 2010*

Bain, L.J. and Engelhardt, M., (1992). *Introduction to probability and mathematical statistics*, *second edition*, *page 292*. Brooks/Cole Cengage Learning

De Boor, C. (1978). *A Practical guide to Splines*, Springer-Verlag, New York.

Draper, N.R. and Smith, H. (1998). *Applied Regression Analysis, Third Edition*, John Wiley & Sons, New York.

European Union (2014). *EU Cohesion Policy 2014-2020.*

Freedman, D.A. (2005). *Statistical Models, Theory and Practice*, Cambridge University Press, New York.

Huang, J.Z. (2003). *Local asymptotics for polynomial spline regression*. The Annals of Statistics, **31**, 5, 1600–1635.

Jennrich, R.I. (1969). *Asymptotic Properties of Non-Linear Least Squares Estimators*. The Annals of Mathematical Statistics, **40**, 2, 633-643.

Jupp, D. (1975). *The Lethargy Theorem-A Property of Approximation by Polynomials*. Journal of Approximation Theory, 14, 204-217.

King, G. (1907) *Notes on summation formulas of graduation, with certain new formulas for consideration*, *J. Inst. Actuar.*, vol. 41, pp. 530–565,.

Lindstrom, M. J. (1999). *Penalized Estimation of Free-Knot Splines*. Journal of Computational and Graphical Statistics, Vol. 8, No. 2, pp.333-352

Mao, W., Zhao L. H. (2003). *Free-Knot Polynomial Splines with Confidence Intervals*. Journal of the Royal Statistical Society. Series B (Statistical Methodology), Vol. 65,No. 4, pp. 901-919

Miyata, S., Shen, X. (2003). *Adaptive Free-Knot Splines*. Journal of Computational and Graphical Statistics, Vol. 12, No. 1, pp.197-213

McCormack, K . (2017). *Graduation of Crude Mortality Rates using a Self-Adaptive B-Spline Model*. Journal of Computational Statistics & Data Analysis. Manuscript in preparation.

McCormack, K . (2017). *Comparison of regional life expectancy curves using Hausdorff and Fréchet distances*. Manuscript in preparation

McCutcheon, J. J. (1975-1977). *Some elementary Life Table approximations*. Transactions of the Faculty of Actuaries 35 , 297-301

Shang, Z. and Cheng, G. (2013). Local and global asymptotic inference in smoothing spline models. *The Annals of Statistics*, **41**, 5. 2608-2638.

Spiriti, S., Eubank, R., Smith, P.W. and Young, D. (2013). *Knot selection for least-squares and penalized splines.* Journal of Statistical Computation and Simulation Vol. 83 , Iss. 6.

Wahba, G. (1983). Bayesian “Confidence Intervals” for the Corss-validated Smoothing Spline. *Journal of the Royal Statistical Society*, **45**, 1, 133-150.

White, H. (1980). *A Heteroskedasticity-Consistent Covariance Matrix Estimator and a Direct Test for Heteroskedasticity*, Econometrica 48 (4), 817-838.

University College Cork (2013). *Variations in the health status of Irish Regions.* Centre for Policy Studies, Working Paper Series CPS WP: 13-009.

Appendix 1

A.1 Glossary of technical terms

x the exact age of the person, that is, on his or her birthday.

l_x the number of persons surviving to exact age x out of the original 100,000 aged 0.

d_x the number of deaths in the year of age x to $x+1$ out of l_x persons who enter that year.

p_x the probability of surviving a year, or the ratio of the number completing the year of age x to $x+1$ to the number entering on the year.

q_x the rate of mortality, the probability of dying in a year, or the ratio of the number of deaths in the year of age x to $x+1$ to the number entering on the year.

L_x the population to be expected according to the Life Table aged between x to $x+1$ years, assuming deaths occur evenly over year.

T_x the expected number of person years to be lived by the survivors at age x.

e_x^0 life expectancy at age x for each person surviving, or the total future life time in years which will on average be passed through by persons aged exactly x.

A.2: Examples of calculations

Using the Male Irish Life Table No. 16, examples of the above terms are as calculated as follows.

The first column of the life table, l_x equals the number of persons surviving in the life table at each exact age x, in other words the January population. l_0 represents the life table population of new born children or those aged exactly zero. If one lets l_0 equal 100,000 then for example, l_5 is the number of persons surviving on their fifth birthday, which in this case equals 99,560.

The second column of the life table, d_x equals the expected number of deaths of persons aged age x in the life table.

$$d_x = l_x - l_{x+1} \qquad \textit{(A.1)}$$

Equation *A.1* tells us that the number of deaths equals the number of persons surviving at age x less the number of persons surviving at age $x + 1$.

For example, for males aged 5 years

$$\begin{aligned} d_5 &= l_5 - l_6 \\ &= 99560 - 99550 \\ &= 10 \end{aligned}$$

The third column of the life table, p_x equals the probability of surviving from exact age x to $x + 1$ and is defined as

$$p_x = 1 - q_x \qquad \textit{(A.2)}$$

For example, for males aged 5 years,

$$p_5 = 1 - 0001026 = 0.9998974,$$

is the probability of surviving to one's fifth year.

The fourth column of the life table, q_x equals the probability of dying between one birthday and the next (See *Section 4*). This may also be called the risk of dying in a life table year, in other words the risk of dying at a particular age. The probability of dying and the probability of survival equal unity. In other words one can only be alive or dead.

$$p_x + q_x = 1 \qquad (A.3)$$

The fifth column of the life table, L_x equals the number of years survived by the life table cohort between the ages x and $x+1$, in other words the July population. Assuming a uniform distribution of deaths over a year of age and using equation *A.1* then:

$$L_x = l_x - \frac{d_x}{2} \qquad (A.4)$$

For example, for Males aged 1 year this means

$$\begin{aligned} L_1 &= l_1 - \frac{d_1}{2} \\ &= 99621 - \frac{35}{2} \\ &= 99604 \end{aligned}$$

The sixth column of the life table, T_x equals the total number of years which will be survived at age x, l_x. So if L_x is person years, then T_x is cumulated person years, i.e.

$$T_x = \sum_{x}^{105} L_x \qquad (A.5)$$

For example, for Males aged 102 years

$$T_{102} = L_{102} + L_{103} + L_{104} + L_{105}$$

The final column of the life table, e_x^0 is the life expectancy in years,

$$e_x^0 = \frac{T_x}{l_x} \qquad (A.6)$$

e_x^0 represents life expectancy at birth and it is broadly used to express the level of mortality. Life expectancy is the average number of additional years a person would live if current mortality trends were to continue. The expectation of life at birth represents the mean length of life of individuals who are subjected since birth to current mortality trends. Life expectancy is usually compiled on the basis of a life table showing the probability of dying at each age for a given population according to the age specific death rates prevailing in a given period.

A.3 Further information

The association between the probability of surviving with that of dying is presented in Equation *A.3.* One can, therefore, make assumptions on the probability of surviving from the probability of dying.

The survivorship ratio at age x, S_x is defined as:

$$S_x = \frac{L_x}{L_{x-1}},$$

which is the ratio of those surviving between ages x and $x+1$ and those surviving between the ages $x-1$ and x. For example, the ratio of those aged 5-9 surviving to age 10-14 is calculated as follows:

$$S_{10-14} = \frac{\sum_{10}^{14} L_x}{\sum_5^9 L_x}$$

Similarly, the probability of a man aged 20 dying before his 50th birthday is calculated as follows:

$$
\begin{aligned}
q_x &= 1 - p_x \\
&= 1 - \frac{l_{x+1}}{l_x} \\
&= \frac{l_x - l_{x+1}}{l_x}
\end{aligned}
$$

therefore,

$$
\begin{aligned}
q_{20-50} &= \frac{l_{20} - l_{50}}{l_{20}} \\
&= \frac{99237 - 95409}{99237} \\
&= 0.0385 \\
&= 3.9\%
\end{aligned}
$$

Appendix 2: Life Tables

Table A2.1: Male Life Table, B-Spline- 2011

Age x	l_x	d_x	p_x	q_x	L_x	T_x	e_x^0
0	100000	379	0.9962099	0.00379014	99,810	7,836,763	78.37
1	99621	35	0.9996508	0.00034918	99,604	7,736,953	77.66
2	99586	11	0.9998916	0.00010843	99,581	7,637,349	76.69
3	99575	7	0.9999264	0.00007365	99,572	7,537,768	75.70
4	99568	8	0.9999177	0.00008227	99,564	7,438,197	74.70
5	99560	10	0.9998974	0.00010263	99,555	7,338,633	73.71
6	99550	11	0.9998880	0.00011196	99,544	7,239,078	72.72
7	99539	11	0.9998914	0.00010860	99,533	7,139,534	71.73
8	99528	10	0.9999019	0.00009812	99,523	7,040,001	70.73
9	99518	9	0.9999135	0.00008653	99,514	6,940,478	69.74
10	99509	8	0.9999220	0.00007805	99,505	6,840,964	68.75
11	99502	8	0.9999246	0.00007543	99,498	6,741,459	67.75
12	99494	8	0.9999181	0.00008186	99,490	6,641,961	66.76
13	99486	10	0.9998955	0.00010451	99,481	6,542,471	65.76
14	99476	16	0.9998435	0.00015650	99,468	6,442,990	64.77
15	99460	23	0.9997670	0.00023301	99,448	6,343,523	63.78
16	99437	33	0.9996713	0.00032866	99,420	6,244,074	62.79
17	99404	44	0.9995598	0.00044020	99,382	6,144,654	61.81
18	99360	56	0.9994389	0.00056113	99,332	6,045,272	60.84
19	99305	68	0.9993177	0.00068233	99,271	5,945,939	59.88
20	99237	79	0.9992067	0.00079329	99,197	5,846,669	58.92
21	99158	88	0.9991161	0.00088386	99,114	5,747,471	57.96
22	99070	94	0.9990541	0.00094587	99,024	5,648,357	57.01
23	98977	96	0.9990255	0.00097449	98,929	5,549,333	56.07
24	98880	96	0.9990312	0.00096877	98,832	5,450,405	55.12
25	98785	92	0.9990686	0.00093142	98,738	5,351,572	54.17
26	98692	86	0.9991319	0.00086807	98,650	5,252,834	53.22
27	98607	79	0.9992012	0.00079885	98,567	5,154,184	52.27
28	98528	76	0.9992251	0.00077489	98,490	5,055,617	51.31
29	98452	78	0.9992044	0.00079558	98,413	4,957,127	50.35
30	98373	81	0.9991724	0.00082761	98,333	4,858,714	49.39
31	98292	85	0.9991367	0.00086332	98,250	4,760,382	48.43
32	98207	89	0.9990970	0.00090302	98,163	4,662,132	47.47
33	98118	93	0.9990529	0.00094708	98,072	4,563,969	46.51
34	98025	98	0.9990041	0.00099589	97,977	4,465,897	45.56
35	97928	103	0.9989501	0.00104990	97,876	4,367,921	44.60
36	97825	109	0.9988904	0.00110961	97,771	4,270,044	43.65

Table A2.1: Male Life Table, B-Spline- 2011 **ctd.**

Age x	l_x	d_x	p_x	q_x	L_x	T_x	e_x^0
37	97717	115	0.9988244	0.00117560	97,659	4,172,273	42.70
38	97602	122	0.9987515	0.00124849	97,541	4,074,614	41.75
39	97480	130	0.9986710	0.00132901	97,415	3,977,074	40.80
40	97350	138	0.9985820	0.00141796	97,281	3,879,659	39.85
41	97212	147	0.9984838	0.00151623	97,138	3,782,377	38.91
42	97065	158	0.9983752	0.00162485	96,986	3,685,239	37.97
43	96907	169	0.9982551	0.00174493	96,823	3,588,253	37.03
44	96738	182	0.9981222	0.00187776	96,647	3,491,430	36.09
45	96556	196	0.9979752	0.00202476	96,459	3,394,783	35.16
46	96361	211	0.9978125	0.00218754	96,255	3,298,325	34.23
47	96150	228	0.9976321	0.00236788	96,036	3,202,069	33.30
48	95922	246	0.9974322	0.00256781	95,799	3,106,033	32.38
49	95676	267	0.9972104	0.00278959	95,543	3,010,234	31.46
50	95409	290	0.9969642	0.00303577	95,264	2,914,691	30.55
51	95120	315	0.9966908	0.00330920	94,962	2,819,427	29.64
52	94805	343	0.9963869	0.00361308	94,633	2,724,465	28.74
53	94462	373	0.9960490	0.00395103	94,276	2,629,831	27.84
54	94089	407	0.9956729	0.00432707	93,885	2,535,556	26.95
55	93682	445	0.9952542	0.00474576	93,460	2,441,670	26.06
56	93237	486	0.9947878	0.00521218	92,994	2,348,211	25.19
57	92751	532	0.9942679	0.00573206	92,485	2,255,217	24.31
58	92220	582	0.9936882	0.00631181	91,929	2,162,731	23.45
59	91638	638	0.9930414	0.00695864	91,319	2,070,802	22.60
60	91000	699	0.9923194	0.00768060	90,650	1,979,484	21.75
61	90301	766	0.9915132	0.00848676	89,918	1,888,833	20.92
62	89535	840	0.9906128	0.00938724	89,114	1,798,916	20.09
63	88694	922	0.9896066	0.01039338	88,233	1,709,801	19.28
64	87772	1011	0.9884821	0.01151787	87,267	1,621,568	18.47
65	86761	1108	0.9872251	0.01277488	86,207	1,534,301	17.68
66	85653	1215	0.9858197	0.01418025	85,046	1,448,094	16.91
67	84438	1330	0.9842484	0.01575163	83,773	1,363,048	16.14
68	83108	1455	0.9824913	0.01750868	82,381	1,279,275	15.39
69	81653	1590	0.9805267	0.01947326	80,858	1,196,894	14.66
70	80063	1735	0.9783303	0.02166966	79,196	1,116,036	13.94
71	78328	1890	0.9758752	0.02412482	77,383	1,036,840	13.24
72	76439	2054	0.9731315	0.02686850	75,412	959,457	12.55
73	74385	2227	0.9700664	0.02993361	73,271	884,045	11.88
74	72158	2407	0.9666436	0.03335637	70,955	810,774	11.24
75	69751	2593	0.9628235	0.03717654	68,455	739,819	10.61

Table A2.1: Male Life Table, B-Spline- 2011 **ctd.**

Age x	*lx*	*dx*	*px*	*qx*	*Lx*	*Tx*	e_x^0
76	67158	2783	0.9585623	0.04143771	65,767	671,364	10.00
77	64375	2973	0.9538126	0.04618738	62,889	605,598	9.41
78	61402	3161	0.9485227	0.05147726	59,822	542,709	8.84
79	58241	3341	0.9426367	0.05736325	56,571	482,887	8.29
80	54900	3508	0.9360944	0.06390562	53,146	426,317	7.77
81	51392	3657	0.9288311	0.07116888	49,563	373,171	7.26
82	47734	3782	0.9207783	0.07922174	45,844	323,608	6.78
83	43953	3874	0.9118632	0.08813676	42,016	277,764	6.32
84	40079	3927	0.9020100	0.09799000	38,115	235,748	5.88
85	36152	3935	0.8911396	0.10886036	34,184	197,633	5.47
86	32216	3893	0.8791713	0.12082873	30,270	163,449	5.07
87	28323	3795	0.8660231	0.13397686	26,426	133,180	4.70
88	24529	3640	0.8516141	0.14838593	22,709	106,754	4.35
89	20889	3429	0.8358652	0.16413475	19,175	84,045	4.02
90	17460	3166	0.8187025	0.18129754	15,878	64,870	3.72
91	14295	2858	0.8000588	0.19994125	12,866	48,992	3.43
92	11437	2517	0.7798775	0.22012245	10,178	36,127	3.16
93	8919	2157	0.7581164	0.24188361	7,841	25,949	2.91
94	6762	1794	0.7347512	0.26524880	5,865	18,108	2.68
95	4968	1442	0.7097812	0.29021876	4,247	12,243	2.46
96	3526	1117	0.6832348	0.31676523	2,968	7,996	2.27
97	2409	831	0.6551755	0.34482447	1,994	5,028	2.09
98	1579	580	0.6324683	0.36753170	1,288	3,034	1.92
99	998	648	0.6491586	0.35084135	674	1,745	1.75
100	648	452	0.6968287	0.30317131	422	1,071	1.65
101	452	280	0.6197644	0.38023555	312	649	1.44
102	280	76	0.2727366	0.72726337	242	337	1.20
103	76	37	0.4811619	0.51883813	58	95	1.25
104	37	32	0.8630853	0.13691468	21	37	1.02
105	32	30	0.9595670	0.04043304	16	16	0.52

Table A2.2: Female Life Table, B-Spline- 2011

Age x	l_x	d_x	p_x	q_x	L_x	T_x	e_x^0
0	100000	329	0.9967098	0.00329021	99,835	8,273,787	82.74
1	99671	68	0.9993223	0.00067768	99,637	8,173,951	82.01
2	99603	9	0.9999110	0.00008903	99,599	8,074,314	81.06
3	99595	6	0.9999373	0.00006267	99,591	7,974,715	80.07
4	99588	7	0.9999338	0.00006623	99,585	7,875,124	79.08
5	99582	8	0.9999236	0.00007639	99,578	7,775,539	78.08
6	99574	7	0.9999278	0.00007218	99,571	7,675,961	77.09
7	99567	6	0.9999424	0.00005760	99,564	7,576,390	76.09
8	99561	5	0.9999505	0.00004953	99,559	7,476,826	75.10
9	99556	5	0.9999524	0.00004755	99,554	7,377,267	74.10
10	99552	5	0.9999499	0.00005006	99,549	7,277,714	73.10
11	99547	6	0.9999433	0.00005671	99,544	7,178,165	72.11
12	99541	7	0.9999321	0.00006790	99,538	7,078,621	71.11
13	99534	8	0.9999157	0.00008434	99,530	6,979,083	70.12
14	99526	11	0.9998933	0.00010669	99,520	6,879,553	69.12
15	99515	13	0.9998651	0.00013494	99,508	6,780,033	68.13
16	99502	17	0.9998325	0.00016755	99,493	6,680,524	67.14
17	99485	20	0.9997995	0.00020048	99,475	6,581,031	66.15
18	99465	23	0.9997730	0.00022696	99,454	6,481,556	65.16
19	99443	24	0.9997613	0.00023867	99,431	6,382,102	64.18
20	99419	23	0.9997671	0.00023288	99,407	6,282,672	63.19
21	99396	22	0.9997807	0.00021928	99,385	6,183,264	62.21
22	99374	21	0.9997916	0.00020844	99,363	6,083,880	61.22
23	99353	21	0.9997908	0.00020925	99,343	5,984,516	60.23
24	99332	23	0.9997679	0.00023206	99,321	5,885,173	59.25
25	99309	28	0.9997183	0.00028166	99,295	5,785,853	58.26
26	99281	30	0.9997017	0.00029827	99,267	5,686,557	57.28
27	99252	24	0.9997548	0.00024517	99,240	5,587,291	56.29
28	99227	23	0.9997668	0.00023316	99,216	5,488,051	55.31
29	99204	25	0.9997440	0.00025599	99,192	5,388,836	54.32
30	99179	28	0.9997127	0.00028726	99,165	5,289,644	53.33
31	99150	32	0.9996784	0.00032159	99,134	5,190,479	52.35
32	99118	36	0.9996408	0.00035920	99,101	5,091,345	51.37
33	99083	40	0.9995997	0.00040035	99,063	4,992,244	50.38
34	99043	44	0.9995547	0.00044528	99,021	4,893,181	49.40
35	98999	49	0.9995057	0.00049429	98,975	4,794,160	48.43
36	98950	54	0.9994523	0.00054769	98,923	4,695,186	47.45
37	98896	60	0.9993942	0.00060581	98,866	4,596,263	46.48
38	98836	66	0.9993310	0.00066900	98,803	4,497,397	45.50
39	98770	73	0.9992623	0.00073767	98,733	4,398,594	44.53

Table A2.2: Female Life Table, B-Spline- 2011 **ctd.**

Age x	*lx*	*dx*	*px*	*qx*	*Lx*	*Tx*	e_x^0
40	98697	80	0.9991878	0.00081224	98,657	4,299,860	43.57
41	98617	88	0.9991068	0.00089320	98,573	4,201,203	42.60
42	98529	97	0.9990189	0.00098106	98,480	4,102,630	41.64
43	98432	106	0.9989236	0.00107639	98,379	4,004,150	40.68
44	98326	116	0.9988201	0.00117985	98,268	3,905,771	39.72
45	98210	127	0.9987079	0.00129214	98,147	3,807,502	38.77
46	98083	139	0.9985859	0.00141406	98,014	3,709,356	37.82
47	97945	151	0.9984535	0.00154649	97,869	3,611,342	36.87
48	97793	165	0.9983096	0.00169041	97,710	3,513,473	35.93
49	97628	180	0.9981531	0.00184695	97,538	3,415,762	34.99
50	97447	197	0.9979827	0.00201734	97,349	3,318,225	34.05
51	97251	214	0.9977970	0.00220300	97,144	3,220,876	33.12
52	97037	233	0.9975945	0.00240550	96,920	3,123,732	32.19
53	96803	254	0.9973733	0.00262665	96,676	3,026,812	31.27
54	96549	277	0.9971315	0.00286847	96,411	2,930,136	30.35
55	96272	302	0.9968667	0.00313326	96,121	2,833,725	29.43
56	95970	329	0.9965764	0.00342364	95,806	2,737,604	28.53
57	95642	358	0.9962574	0.00374259	95,463	2,641,798	27.62
58	95284	390	0.9959065	0.00409351	95,089	2,546,335	26.72
59	94894	425	0.9955197	0.00448027	94,681	2,451,246	25.83
60	94469	464	0.9950927	0.00490733	94,237	2,356,565	24.95
61	94005	506	0.9946202	0.00537978	93,752	2,262,328	24.07
62	93499	552	0.9940965	0.00590350	93,223	2,168,576	23.19
63	92947	603	0.9935148	0.00648522	92,646	2,075,353	22.33
64	92345	659	0.9928673	0.00713274	92,015	1,982,707	21.47
65	91686	720	0.9921449	0.00785509	91,326	1,890,691	20.62
66	90966	788	0.9913373	0.00866270	90,572	1,799,365	19.78
67	90178	863	0.9904323	0.00956770	89,746	1,708,794	18.95
68	89315	945	0.9894158	0.01058422	88,842	1,619,047	18.13
69	88370	1036	0.9882713	0.01172873	87,851	1,530,205	17.32
70	87333	1137	0.9869795	0.01302048	86,765	1,442,354	16.52
71	86196	1248	0.9855180	0.01448204	85,572	1,355,589	15.73
72	84948	1371	0.9838601	0.01613991	84,262	1,270,017	14.95
73	83577	1506	0.9819747	0.01802525	82,823	1,185,755	14.19
74	82070	1656	0.9798252	0.02017484	81,242	1,102,932	13.44
75	80414	1820	0.9773679	0.02263211	79,504	1,021,689	12.71
76	78594	2000	0.9745515	0.02544851	77,594	942,185	11.99
77	76594	2197	0.9713149	0.02868506	75,496	864,591	11.29
78	74397	2412	0.9675857	0.03241428	73,191	789,095	10.61
79	71986	2643	0.9632775	0.03672246	70,664	715,903	9.95

Table A2.2: Female Life Table, B-Spline- 2011 **ctd.**

Age x	*lx*	*dx*	*px*	*qx*	*Lx*	*Tx*	e_x^0
80	69342	2892	0.9582954	0.04170456	67,896	645,239	9.31
81	66450	3153	0.9525514	0.04744857	64,874	577,343	8.69
82	63297	3421	0.9459559	0.05404413	61,587	512,469	8.10
83	59877	3687	0.9384176	0.06158239	58,033	450,882	7.53
84	56189	3942	0.9298475	0.07015253	54,218	392,849	6.99
85	52247	4171	0.9201629	0.07983711	50,162	338,631	6.48
86	48076	4361	0.9092936	0.09070642	45,896	288,469	6.00
87	43715	4494	0.8971883	0.10281166	41,468	242,574	5.55
88	39221	4557	0.8838228	0.11617723	36,943	201,106	5.13
89	34664	4534	0.8692074	0.13079261	32,397	164,163	4.74
90	30130	4417	0.8533959	0.14660408	27,922	131,766	4.37
91	25713	4204	0.8364927	0.16350730	23,611	103,844	4.04
92	21509	3900	0.8186587	0.18134133	19,559	80,233	3.73
93	17608	3520	0.8001149	0.19988514	15,849	60,674	3.45
94	14089	3083	0.7811427	0.21885730	12,547	44,826	3.18
95	11005	2618	0.7620805	0.23791952	9,696	32,279	2.93
96	8387	2153	0.7433158	0.25668423	7,311	22,582	2.69
97	6234	1713	0.7252739	0.27472613	5,378	15,272	2.45
98	4521	1318	0.7084032	0.29159681	3,862	9,894	2.19
99	3203	2220	0.6930682	0.30693179	2,093	6,032	1.88
100	2220	1502	0.6767526	0.32324744	1,469	3,939	1.77
101	1502	981	0.6531123	0.34688766	1,012	2,470	1.64
102	981	603	0.6142550	0.38574501	680	1,458	1.49
103	603	331	0.5495238	0.45047618	437	778	1.29
104	331	184	0.5540559	0.44594407	239	341	1.03
105	184	163	0.8898570	0.11014299	102	102	0.56