ORBITAL ECCENTRICITY AND ITS EFFECT ON STABILITY

Introduction

With the development of satellite constellations, it has become increasingly necessary to simulate vast quantities of orbits and compare their relative stabilities to determine the optimal choice. However, there is no formula to do this, and current methods are complex and inefficient, making them unsuitable. I seek to define an efficient index which both quantifies and categorises stability, given just one variable, eccentricity, with a minimal computational workload. Eccentricity measures how elliptical the path the orbit follows is, as shown in the diagram below.

Aims

- Determine the stability of an orbit of given eccentricity.
- Evaluate whether a correlation exists between eccentricity and stability.
- Formulate a scale that assigns a value of stability to a given orbit.
- Break this scale into descriptive categories that classify stability.

Data Generation

The three variables defining an orbit are the semi-major axis (orbit dimensions), the central body (gravitational force), and the eccentricity (orbit shape). The system was simplified to two variables by calculating a representative sample of 100 orbits varying from eccentricity of 0 to 0.99, for each combination of the other two parameters. One variable was changed at a time, analysing all planetary orbits in the solar system: first, changing the semi-major axis with the Sun as a constant body; then, changing the central body to each planet while maintaining a constant semi-major axis. 3 parameters were calculated independently in each case, explained in detail later. 1,800 orbits were simulated, split into 18 groups of 100 each; 9 for varying central body and 9 for varying axes.

Method

1. Firstly, I defined stability as resistance to perturbation, or simply how much force the orbit can take before the body breaks free. I selected the 3 parameters associated with this condition and calculated them independently by simulating each orbit in MATLAB R2016a.
2. I transformed the data into a general set of values that apply to all possible cases, reducing the dependent variable to eccentricity. ANOVAs and Linear Regression were used to interpret the data, and then PCA was applied to reduce dimensionality from 18 sets to just 1 for each parameter. Chi Square Goodness-of-Fit tests verified correlation.
3. The datasets were standardized, and Multiple Regression Analysis was performed on these Z-scores to generate regression coefficients and determine each variable's relative importance. I used the Ordinary Least Squares Method, where $\beta = (X^T X)^{-1} X^T Y$. Equal weighting was assumed to generate values for the dependent variable of stability, and the coefficients were produced from this data.
4. Regression coefficients were normalized and defined as weights for each variable. Values of stability ranging from 0 to 1 were calculated for each eccentricity by summing the products of normalized regression coefficients and their associated variables for each eccentricity.
5. To descriptively categorise this data, piecewise linear regression was used, introducing two breakpoints. Initial guesses were derived from k means clustering, and optimized by iteratively minimizing Residual Sum of Squares (RSS).

I coded a MATLAB R2016a program to propagate all 1,800 orbit simulations. I imported the associated data to Excel spreadsheets, where it was analysed and used to generate graphs.

Parameters Assessed

1. **Escape Velocity:** This is the velocity required for the body to escape the gravity of the planet.
2. **Delta Velocity (Delta V):** This refers to the extra velocity required to break the orbit, and allow the object to escape the gravity of the planet. It is the difference between the velocity at a given point and Escape Velocity.
3. **Energy Distribution:** This refers to the ratio of energy stored as potential or kinetic energy. The lowest, and therefore most stable percentage possible is 50%, at eccentricity of 0, as the orbit is circular.

Statistical Analysis

I ran an ANOVA on each of the 3 parameters, analysing the 18 datasets of each to test the following hypothesis, where $\alpha = 0.05$:

$H_0 : \mu_{Earth} = \mu_{Mars} \cdots \mu_{Pluto}$ (For all 18 simulations)
H_A: The means are not equal

Observed P-values for Escape Velocity and Delta V reject H_0, indicating significant statistical difference. There was no difference between mean energy ratios.

Table 1:

ANOVAs		
Metric	*F*	*P-value*
Escape Velocity	197.57	0.00*
Delta V	679.64	0.00*
Energy Ratio	0	1.00*

I fitted a Linear Regression model to the two variables that rejected the null hypothesis, Escape Velocity and Delta V. The R^2 value and Pearson Coefficient of 1 indicate perfect positive linear correlation.

Table 2:

Linear Regression Model		
Metric	R^2	*Pearson*
Escape Velocity	1.00	1.00*
Delta V	1.00	1.00*

I ran PCA on these sets, resulting in two constant covariance matrices of 1, proving equal weighting of all dimensions. I reduced all datapoints to one representative set by multiplying the 18 individual values for each eccentricity by the Eigenvector for PC1 and summing the products.

To verify the correlation of these general sets to the raw values, I used them as standards, and applied Chi-Square Goodness-of-Fit tests to each variable. The resultant p-values were all 1, validating perfect correlation to the general model. The line graph below shows a randomly selected set of values against the standard, and the equal R^2 values of each trendline.

Table 3:

Principal Component Analysis		
Metric	Escape Velocity	Delta V
Convariance p-value	1*	1*
Number of Components	18	18
Significant Components	PC1*	PC1*
PC1 Eigenvalue	18	18
PC1 Eigenvector	0.24	0.24

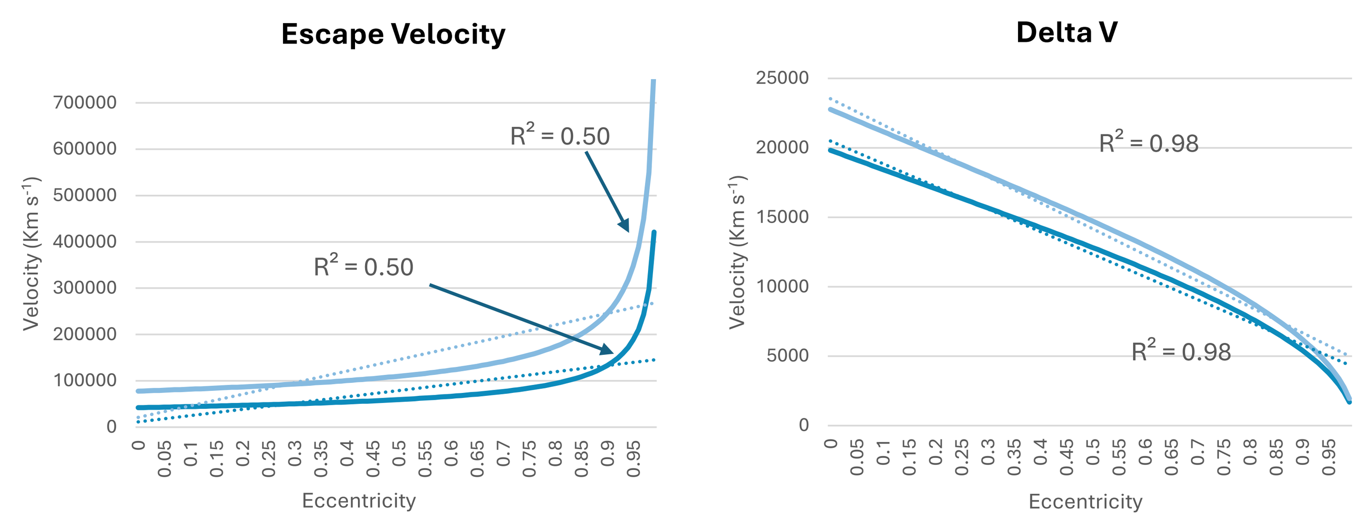

Fig. 1: A randomly selected dataset from each parameter plotted against the general dataset obtained from PCA.

NORMALIZED VALUES AGAINST ECCENTRICITY

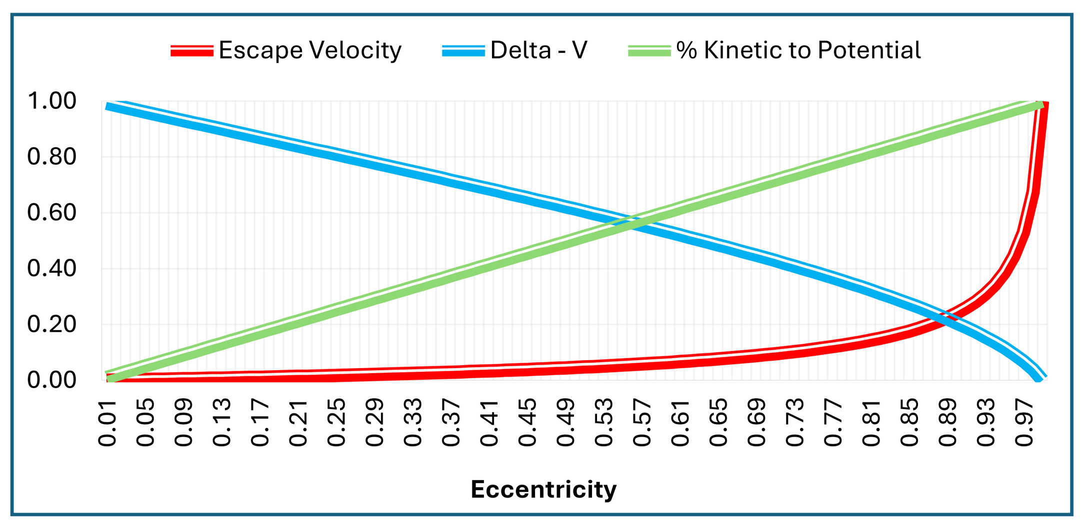

Fig. 2: Normalized Escape Velocity, Delta V and Energy Distribution against Eccentricity.

The accompanying graph is the set of normalized values of the general datasets, plotted against the corresponding eccentricity. Varying R^2 Values and the curved shape indicate that not all quantities are linearly related to eccentricity. Clearly, the stability index must account for this fluctuation. There is a tendency toward extremity at high eccentricities, indicating a rapid decrease in stability above ~ 0.85.

The coefficients from Multiple Regression Analysis on the Z-scores of the above variables are shown on the right. These quantify the relative importance of each factor, and negate the assumption of equal weighting.

Clearly, eccentricity is the most significant variable, with a relative importance of 52%.

Table 4:

Multiple Regression Analysis	
Parameter	Coefficient
Eccentricity	1.00
Escape Velocity	0.15
Delta V	0.27
Energy Distribution	-0.50

NORMALIZED WEIGHT

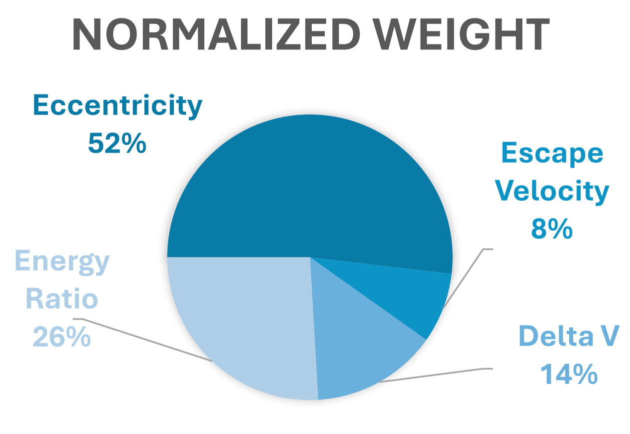

Fig. 3: Pie chart comparing the normalized Regression Coefficients for each variable.

Defining a Scale

I selected the parameter with the highest standard deviation, Escape Velocity, to break the data into sections. K-means clustering was initially performed; however, the inflection point in the elbow graph at k=1 indicates it is not a suitable method.

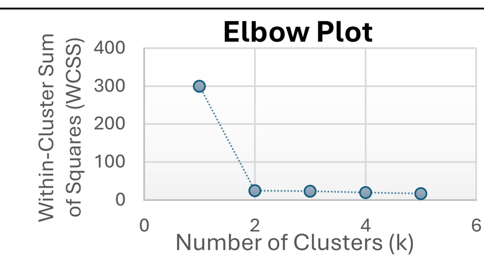

Fig. 4: Elbow plot for k mean clustering

Using piecewise linear regression, two breakpoints were introduced, defining the initial guesses as half the distance between centroids of the earlier k=3. An iterative optimization approach using Excel Solver defined the most suitable breakpoints by minimizing Residual Sum of Squares (RSS).

Table 5:

Piecewise Linear Regression	
Groups	3
Breakpoint 1	0.5
Breakpoint 2	0.91
F Statistic	24912.43
p-value	2.95E-141

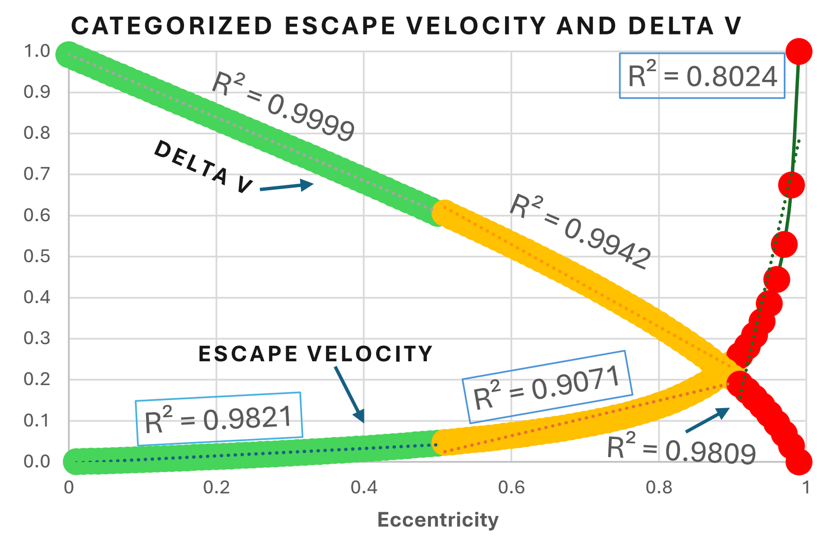

Fig. 5: Normalized Delta V and Escape Velocity broken into categories with corresponding R^2 values against Eccentricity

Linear regression models were fitted to each segment, and an F-test was performed to compare the piecewise model against a single linear regression model. The resulting F-statistic and p-value validated a statistically significant improvement, where $\alpha = 0.05$.

These segments are stability categories, and represent the optimal splitting of the curve into three parts, resulting in the highest possible R^2 values for each segment.

Table 6:

Stability Classification	
Descriptor	Eccentricity
Stable	$0 \le e < 0.5$
Moderately Stable	$0.5 < e < 0.91$
Unstable	$0.91 < e < 1$

Stability Thresholds

I generated values of stability, resulting in a scale from 0-1, where the most stable orbit (at eccentricity of 0) is equal to 1. I calculated the Pearson and Spearman Rank Coefficient between Eccentricity and Stability. It indicates a perfect negative monotonic relationship that occurs at an inconsistent rate. Though extremely close, the Pearson Coefficient is not perfect.

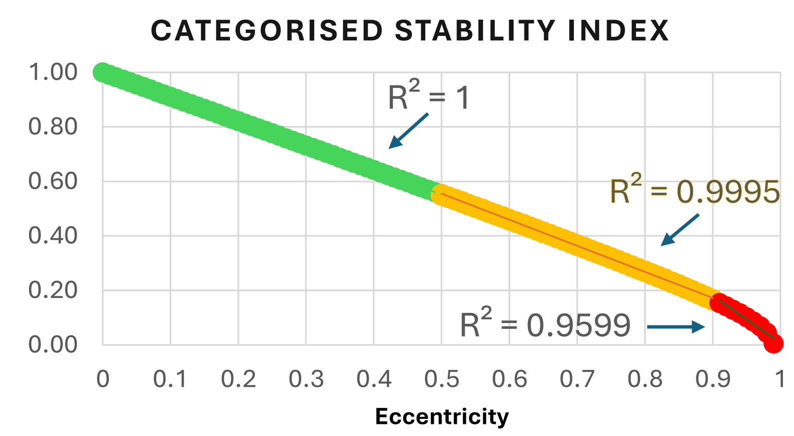

Fig. 6: Categorised Stability Index with R^2 values against Eccentricity

Table 7:

Descriptive Statistics of Each Category					
Metric	Standard Deviation	Mean	Eccentricities in Range	R^2	Pearson
Stable	0.11	0.82	50%	1.00	-1.00
Moderately Stable	0.10	0.47	41%	0.9995	-0.998
Unstable	0.08	0.17	9%	0.96	-0.95

Table 8:

Overall Correlation	
Spearman	Pearson
-1	-0.999

Were these values to be rounded up, they would lose significance.

Conclusion

In conclusion, eccentricity is very closely related to stability. Fluctuating Pearson Coefficients between stability descriptors show a degeneration in correlation as eccentricity increases. The distribution of datapoints along the stability descriptors shows that 50% of possible orbits lie within the stable range, 41% are moderately stable, and 9% are unstable.

I successfully defined a scale that assigns a categorised value of stability to an orbit, taking into account the relative significance of each contributing factor. This index applies to the set of all possible orbits, regardless of size, or the central body. The variance in stability has been structured to be uniquely dependent on eccentricity. Eccentricity is the most important variable defining the values on the scale, with a coefficient of 52%.

Although comprehensive in its analysis, I believe this model could benefit from factoring in external gravitational forces, and including a more thorough definition of stability.

The stability index generated is efficient, and suitable for analysis of distant constellations and satellite insertion, as it comprehensively classifies stability with extremely limited computational workload, balancing practicality with accuracy.